Solutions to Worksheet 20

a) $SA = \int_0^4 2\pi y \, ds = \int_0^4 2\pi x \sqrt{1+1^2}\, dx$

$= \frac{2\sqrt{2}\pi x^2}{2}\Big|_0^4$

$= \boxed{16\sqrt{2}\,\pi}$

b) $SA = \int_a^b 2\pi y \, ds$

$= \int_0^2 2\pi x^3 \sqrt{1+(3x^2)^2}\, dx$

$u = 1+9x^4$ $\quad x=0: u=1$

$du = 36x^3\,dx$ $\quad x=2: u=145$

$= \int_1^{145} \frac{2\pi}{36}\sqrt{u}\, du$

$= \frac{\pi}{18}\cdot\frac{2}{3}u^{3/2}\Big|_1^{145}$

$= \boxed{\frac{\pi}{27}\left(145^{3/2}-1\right)}$

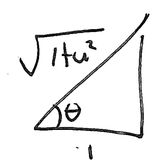
c) $SA = \int_a^b 2\pi x \, ds$ $\qquad \frac{dx}{dy} = -e^{-y}$

$= \int_0^1 2\pi e^{-y}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_0^1 2\pi e^{-y}\sqrt{1+e^{-2y}}\, dy$ $\qquad u = e^{-y}$, $du = -e^{-y}dy$

$= -\int_1^{e^{-1}} 2\pi\sqrt{1+u^2}\, du$

(note this is equivalent to choosing to write ds in terms of dx since $u=x$)



Let $u=\tan\theta$, $du = \sec^2\theta\, d\theta$, then $1+u^2 = 1+\tan^2\theta = \sec^2\theta$

$= \int_{u=e^{-1}}^{u=1} 2\pi\sqrt{\sec^2\theta}\,\sec^2\theta\, d\theta$

$= 2\pi\int_{u=e^{-1}}^{u=1} \sec^3\theta\, d\theta$

$= \pi\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right]\Big|_{u=e^{-1}}^{u=1}$

$= \pi\left[\sqrt{1+u^2}\cdot u + \ln\left|\sqrt{1+u^2}+u\right|\right]\Big|_{e^{-1}}^{1}$

$= \boxed{\pi\left[\sqrt{2} + \ln(\sqrt{2}+1) - \sqrt{1+e^{-2}} - \ln\left(\sqrt{1+e^{-2}}+e^{-1}\right)\right]}$

d) $SA = \int_a^b 2\pi y\, ds$

Write ds in terms of x b/c $y=(4-x^{2/3})^{3/2}$ is annoying to solve for y:

$SA = \int_0^8 2\pi (4-x^{2/3})^{3/2}\sqrt{1+(y')^2}\, dx$

$y' = \frac{3}{2}(4-x^{2/3})^{1/2}\cdot\left(-\frac{2}{3}x^{-1/3}\right)$

$= -\frac{\sqrt{4-x^{2/3}}}{x^{1/3}}$

$SA = \int_0^8 2\pi(4-x^{2/3})^{3/2}\sqrt{1+\frac{4-x^{2/3}}{x^{2/3}}}\, dx$

$= \int_0^8 2\pi(4-x^{2/3})^{3/2}\frac{1}{x^{1/3}}\sqrt{x^{2/3}+4-x^{2/3}}\, dx$

$= 4\pi\int_0^8 (4-x^{2/3})^{3/2}\cdot\frac{1}{x^{1/3}}\, dx$ $\qquad u = 4-x^{2/3}$, $du = -\frac{2}{3}x^{-1/3}dx$ $\qquad x=0: u=4$, $x=8: u=0$

$= 4\pi\cdot\left(-\frac{3}{2}\right)\int_4^0 u^{3/2}\, du$

$= 6\pi\int_0^4 u^{3/2}\, du$

$= 6\pi\left(\frac{2}{5}\right)u^{5/2}\Big|_0^4 = \frac{6\cdot 2\cdot 32}{5}\pi = \boxed{\frac{384\pi}{5}}$

e) $SA = \int_a^b 2\pi y \, ds = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi \int_1^e \left[\frac{1}{4}x^2 - \frac{1}{2}\ln(x)\right]\sqrt{\frac{1}{4}(x+x^{-1})^2}\,dx$

$f'(x) = \frac{1}{2}x - \frac{1}{2}x^{-1}$

$(f'(x))^2 = \frac{1}{4}(x^2 - 2 + x^{-2})$

$1 + (f'(x))^2 = 1 + \frac{1}{4}(x^2 - 2 + x^{-2})$

$= \frac{1}{4}(x^2 - 2 + 4 + x^{-2})$

$= \frac{1}{4}(x^2 + 2 + x^{-2})$

$= \frac{1}{4}(x + x^{-1})^2$

$= \pi \int_1^e \left(\frac{1}{2}x^2 - \ln(x)\right)\frac{(x + x^{-1})}{2}\,dx$

$= \frac{\pi}{2}\int_1^e \frac{1}{2}x^3 - x\ln(x) + \frac{1}{2}x - \frac{\ln(x)}{x}\,dx$

$= \frac{\pi}{2}\left(\frac{1}{8}x^4 - \underbrace{\left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)}_{\substack{\text{by parts} \\ u=\ln(x)\ dv = x\,dx}} + \frac{1}{4}x^2 - \frac{(\ln(x))^2}{2}\right)\Bigg|_1^e$

$= \frac{\pi}{2}\left(\left(\frac{e^4}{8} - \left(\frac{e^2}{2}\cdot 1 - \frac{e^2}{4}\right) + \frac{e^2}{4} - \frac{1^2}{2}\right) - \left(\frac{1}{8} - \left(\frac{1}{2}\cdot 1 \cdot 0 - \frac{1}{4}\right) + \frac{1}{4} - 0\right)\right)$

$= \frac{\pi}{2}\left(\frac{e^4}{8} - \frac{e^2}{4} + \frac{e^2}{4} - \frac{1}{2} - \frac{1}{8} - \frac{1}{4} - \frac{1}{4}\right)$

$= \frac{\pi}{2}\left(\frac{e^4}{8} - \frac{9}{8}\right)$

$= \boxed{\frac{\pi}{16}(e^4 - 9)}$

f) $SA = \int_a^b 2\pi y \, ds$

$= \int_0^\pi 2\pi \sin(x)\sqrt{1 + \cos^2(x)}\,dx$

$= \int_0^\pi 2\pi \sin(x)\cdot\sin(x)\,dx$

$= 2\pi \int_0^\pi \sin^2(x)\,dx$

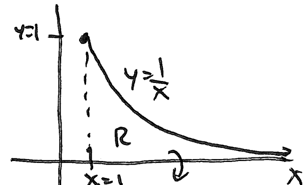
$= 2\pi \int_0^\pi \frac{1}{2}(1 - \cos(2x))\,dx$

$= \pi\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^\pi$

$= \pi\left(\pi - \frac{1}{2}\cancelto{0}{\sin(2\pi)} - 0 + 0\right)$

$= \boxed{\pi^2}$

2)



$V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \int_1^\infty \pi \frac{1}{x^2}\,dx$

$= \lim_{R\to\infty} \int_1^R \pi \frac{1}{x^2}\,dx$

$= \lim_{R\to\infty} \pi\left(-\frac{1}{x}\Big|_1^R\right)$

$= \lim_{R\to\infty} \pi\left(-\frac{1}{R} + 1\right)$

$= \boxed{\pi}$

$SA = \int_1^\infty 2\pi y\sqrt{1 + (y')^2}\,dx$

$= \int_1^\infty 2\pi \frac{1}{x}\sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\,dx$

$= \int_1^\infty \frac{2\pi}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx$

$= 2\pi \int_1^\infty \frac{1}{x^3}\sqrt{x^4+1}\,dx$

$> 2\pi \int_1^\infty \frac{\sqrt{x^4}}{x^3}\,dx$

$= 2\pi \int_1^\infty \frac{1}{x}\,dx$

diverges (p-integral w/ p=1)

So the surface area is ∞.